Section I: Apportionment

Part A: The History of Apportionment

The following is a quote from the U.S. Constitution, Article I, Section 2, which was amended by Amendment XIV (Passed by Congress June 13, 1866. Ratified July 9, 1868):

> *Representatives shall be apportioned among the several States according to their respective numbers, counting the whole number of persons in each State, excluding Indians not taxed. But when the right to vote at any election for the choice of electors for President and Vice-President of the United States, Representatives in Congress, the Executive and Judicial officers of a State, or the members of the Legislature thereof, is denied to any of the male inhabitants of such State, being twenty-one years of age, and citizens of the United States, or in any way abridged, except for participation in rebellion, or other crime, the basis of representation therein shall be reduced in the proportion which the number of such male citizens shall bear to the whole number of male citizens twenty-one years of age in such State.*

The apportionment of representatives means that the number of representatives from each state within Congress is based on its population. In our Electoral College, each state is given the same number of electoral votes as its number of representatives and senators.

The apportionment process is quite complex and has a long and jaded past. In fact, the first presidential veto was issued by George Washington in 1792 because he did not approve of the way the House of Representatives had decided to apportion the number of representatives for each state. The original plan was a plan set forth by Alexander Hamilton. It is referred to as the Hamilton plan. An alternative plan, set forth by Thomas Jefferson (referred to as the Jefferson plan), was eventually accepted.

Over the years, different plans of apportionment have been used over the years, some resulting in direct violations of the U.S. Constitution. A method developed in 1832 by Daniel Webster was similar to Jefferson's but he used advice from mathematicians to help guide his method. However, from 1940 to the present, the Huntington-Hill Method or the Method of equal proportions has been used.

(There are spreadsheet programs at http://college.hmco.com/mathematics/aufmann/excursions/2e/resources/es.html for each of the Hamilton, Jefferson, and Huntington-Hill apportionment plans.)

The following is a complete list of electoral votes for each state over the last three decades.

State	***1981-1990***	***1991-2000***	***2001-2010***	***State***	***1981-1990***	***1991-2000***	***2001-2010***
Alabama	9	9	9	Montana	4	3	3
Alaska	3	3	3	Nebraska	5	5	5
Arizona	7	8	10	Nevada	4	4	5
Arkansas	6	6	6	New Hampshire	4	4	4
California	47	54	55	New Jersey	16	15	15
Colorado	8	8	9	New Mexico	5	5	5
Connecticut	8	8	7	New York	36	33	31
Delaware	3	3	3	North Carolina	13	14	15
D.C.	3	3	3	North Dakota	3	3	3
Florida	21	25	27	Ohio	23	21	20
Georgia	12	13	15	Oklahoma	8	8	7
Hawaii	4	4	4	Oregon	7	7	7
Idaho	4	4	4	Pennsylvania	25	23	21
Illinois	24	22	21	Rhode Island	4	4	4
Indiana	12	12	11	South Carolina	8	8	8
Iowa	8	7	7	South Dakota	3	3	3
Kansas	7	6	6	Tennessee	11	11	11
Kentucky	9	8	8	Texas	29	32	34
Louisiana	10	9	9	Utah	5	5	5
Maine	4	4	4	Vermont	3	3	3
Maryland	10	10	10	Virginia	12	13	13
Massachusetts	13	12	12	Washington	10	11	11
Michigan	20	18	17	West Virginia	6	5	5
Minnesota	10	10	10	Wisconsin	11	11	10
Mississippi	7	7	6	Wyoming	3	3	3
Missouri	11	11	11				

Part B: Hamilton versus Jefferson

In the Hamilton Plan, the total population of the country is divided by the number of total representatives. This gives us the number of citizens represented by each representative. Once this *standard divisor* has been calculated, divide the population of each state by this number and round the quotient down to a whole number. (Both 12.1 and 12.9 would be rounded down to 12.) Each whole number quotient is called a *standard quota*. If the sum of the standard quotas does not add up to the correct number of representatives, then an additional representative is assigned to the states with the largest decimal remainder.

In the Jefferson plan, a *modified standard divisor* is used in an attempt to avoid being short of representatives.

Part C: Fairness in Apportionment

Although fair is a subjective term, there are several criterion by which an apportionment plan is judged to be fair.

1. The Quota Rule

   The *quota rule* states that the number of representatives apportioned to a state is the standard quota or one more than the standard quota.

2. Average Constituency

   $$\text{Average constituency} = \frac{\text{Population of a state}}{\text{Number of representatives from the state}}$$

   The idea of average constituency is an essential part of democracy as each vote in the House of Representatives should have equal weight. Having one vote representing 1000 people and another vote representing 10,000 people would not be equal representation.

3. Absolute Unfairness of an Apportionment

   The *absolute unfairness of an apportionment* is the absolute value of the difference between the average constituency of state A and the average constituency of state B.

   This concept is also used when deciding which investments are most productive or detrimental. Let's look at the following example.

4. Relative Unfairness of an Apportionment

$$\text{Relative unfairness of an apportionment} = \frac{\text{Absolute unfairness of the apportionment}}{\text{Average constituency of the state receiving the new representative}}$$

   The *apportionment principle* states that when adding a new representative to a state, the representative is assigned to the state so as to give the smallest relative unfairness of apportionment.

Part D: Apportionment Paradoxes

1. Alabama Paradox

   This paradox occurs when an increase in the number of total representatives results in a state losing a representative. It was first identified after the 1870 census when Rhode Island lost a representative.

2. Population Paradox

   This paradox occurs when changes in apportionment do not accurately reflect changes in population.

3. New States Paradox

   This paradox occurs with Hamilton's method and appeared when Oklahoma became a state in 1907. When Oklahoma's five seats were added, New York was forced to give up a seat to Maine, despite the fact that no population changes had taken place.

4. Balinski-Young Impossibility Theorem

   No apportionment method is perfect. This was proven by Michael Balinski and H. Peyton Young by violating either the quota rule or by producing a paradox. They presented a strong case of the Webster method because it most closely satisfies the idea of one person, one vote. However, politics prevailed when FDR chose the Huntington-Hill method in 1941 so that his democratic party would have one more seat in the House.

Part E: The Huntington-Hill Apportionment Method *(aka Equal Proportions Method)*

This method is implemented by calculating what is called a Huntington-Hill number, which is derived from the apportionment principle.

P_A = population of state A
P_B = population of state B

a = number of representatives from state A
b = number of representatives from state B

$\frac{P_A}{a+1}$ = average constituency of A when it receives a new representative

$\frac{P_B}{B}$ = average constituency of B without a new representative

The relative unfairness of apportionment by giving A the new member is…

$$\frac{\text{Absolute unfairness}}{\text{Average constituency of A}} = \frac{\frac{P_B}{b} - \frac{P_A}{a+1}}{\frac{P_A}{a+1}}$$

According to the apportionment principle, state A should receive the next representative instead of state B if the relative unfairness to A is less than the relative unfairness to B. That is,

$$\frac{\frac{P_B}{b} - \frac{P_A}{a+1}}{\frac{P_A}{a+1}} < \frac{\frac{P_A}{a} - \frac{P_B}{b+1}}{\frac{P_B}{b+1}}$$

Simplifying the inequality gives…

$$\frac{(P_B)^2}{b(b+1)} < \frac{(P_A)^2}{a(a+1)}$$

The former is the Huntington-Hill number for state B and the later is the number for state A. The state with the largest Huntington-Hill number receives the next representative. This method can be extended to more than two states. This is called the *Huntington-Hill apportionment principle*.

Part F: Apportionment Example

In the country of Math, there are five states: Add, Subtract, Multiply, Divide, and Root. They are setting up their new democratic House of Representatives with twenty-five representatives.

State	Population
Add	11,123
Subtract	879
Multiply	3,518
Divide	1,563
Root	2,917
TOTAL	*20,000*

1. Using the Hamilton Plan, calculate the standard divisor for the country of Math?

2. What is the meaning of this number?

3. Complete the following chart to find the standard quota and number of representatives for each state.

State	Population	Quotient	Standard Quota	Number of Reps
Add	11,123			
Subtract	879			
Multiply	3,518			
Divide	1,563			
Root	2,917			
		TOTAL		

4. Using the Jefferson Plan with a modified standard divisor of 740, complete the chart to find the standard quota and number of representatives for each state.

State	Population	Quotient	Standard Quota	Number of Reps
Add	11,123			
Subtract	879			
Multiply	3,518			
Divide	1,563			
Root	2,917			
		TOTAL		

5. Compare the two plans. Is either of the two plans unfair? If so, what criteria/rule does it violate?

6. If a new representative were to be added, which state would receive it using the Huntington-Hill Principle? (Hint: Calculate the Huntington-Hill number for each state.)

Section II: Voting

Part A: Methods of Voting

1. Majority

   An issue is resolved if more than 50% of the people voting vote for the issue.

2. Plurality Method

   Each voter votes for one candidate, and the candidate with the most first-place votes wins. A majority of votes is not required.

   Voters are often asked to rank each candidate in order of preference. (No ties allowed.) They do this on a *preference ballot*. The results are then grouped in a *preference schedule* where similar ballots are grouped to summarize the voting.

3. Borda Count

   Jean C. Borda (1733 – 1790) was a member of the French Academy of Sciences. He began to contemplate the way in which people were elected to the Academy. He was worried that the plurality method might not result in the best candidate being elected. Borda's method was the first attempt to mathematically quantify voting systems.

   Each place on the ballot is assigned points. In an election with N candidates, we give 1 point for last place, 2 points for next to last place,…, and N points for first place. The pointed are tallied for each candidate and the candidate with the highest total is the winner. The Borda winner is considered the "compromise candidate."

   This method is used for the Heisman Trophy, the American and National Baseball League's MVP, and Country Music Vocalist of the Year. It is also used at some track meets and other events.

4. Plurality with Elimination

   In Round 1 of voting, the first place votes for each candidate are counted. If a candidate has a majority of first-place votes, that candidate is the winner.

   In rounds 2, 3, 4, et cetera, the names of candidates eliminated from the preference schedule are crossed out and then the first-place votes are recounted. The process is repeated until a candidate has a majority of first-place votes.

   This method is used for the mayoral elections in Burlington, Vermont, and modified versions are used to determine Olympic host cities and the winner of the Academy Awards.

5. Pair-wise Comparison (aka Copeland's Method)

   This method is similar to a round-robin tournament in which every candidate is matched one-to-one with every other candidate. The candidate that is preferred over the other candidate gets a point in each match up. The candidate with the most points is the winner.

   A *condorcet candidate* is one who wins in every head-to-head comparison against each of the other candidates.

Part B: Fairness of Voting Methods

1. Fairness Criteria

    a. *Majority criterion*: The candidate who receives a majority of the first-place votes is the winner.

    b. *Monotonicity criterion*: If candidate A wins an election, then candidate A will also win the election if the only change in the voters' preferences is that supporters of a different candidate change their votes to support candidate A.

    c. *Condorcet criterion*: A candidate who wins all possible head-to-head match-ups should win an election when all candidates appear on the ballot.

    d. *Independence of irrelevant alternatives*: If a candidate wins an election, the winner should remain the winner in any recount in which losing candidates withdraw from the race.

2. Arrow's Impossibility Theorem

    Arrow's Impossibility Theorem states that there is no voting method involving three or more choices that satisfies the fairness criteria. Therefore, none of the voting methods discussed previously are not fair.

Part C: Counting the Ballots

This is a very interesting and complex question. Just how do those voting machines count all the ballots that are fed into them? How do the electronic voting machines work? If you are interested in this topic, I would highly recommend viewing the documentary HACKING DEMOCRACY. This documentary was released by HBO in 2007. It is an in-depth look at the Diebold Corporation, which programs and produces the voting machines. The activists involved in this investigation find incendiary evidence of miscounted votes.

For more information about this documentary and the counting of ballots, see www.blackboxvoting.org.

Part D: Voting Example

Students were asked to vote on the band that would play at the annual Metric Day Dance: Binary Code (B), Wiser than Euclid (W), Get Real (G), Sum of Excellence (S), or Radical Roots (R) They used a preference ballot and came up with the following preference schedule.

Number of Voters	18	12	10	9	4	2
1st choice	B	W	G	S	R	R
2nd choice	S	R	W	G	W	G
3rd choice	R	S	R	R	S	S
4th choice	G	G	S	W	G	W
5th choice	W	B	B	B	B	B

1. How many people voted?

2. How many votes would be needed for a majority?

3. Using plurality, who is the winner?

4. Using a Borda count, who is the winner?

5. Using plurality-with-elimination, who is the winner?

6. Using the method of pair-wise comparisons, who is the winner?

7. Do any of these methods violate the majority criterion?

8. Do any of these methods violate the condorcet criterion?

Section III: Answer Key to Problems

Part F: Apportionment Example (KEY)

In the country of Math, there are five states: Add, Subtract, Multiply, Divide, and Root. They are setting up their new democratic House of Representatives with twenty-five representatives.

State	**Population**
Add	*11,123*
Subtract	*879*
Multiply	*3,518*
Divide	*1,563*
Root	*2,917*
TOTAL	*20,000*

1. *Using the Hamilton Plan, calculate the standard divisor for the country of Math?*

$$\frac{20{,}000}{25} = 800$$

2. *What is the meaning of this number?*

It is the number of citizens represented by each representative.

3. *Complete the following chart to find the standard quota and number of representatives for each state.*

State	**Population**	**Quotient**	**Standard Quota**	**Number of Reps**
Add	*11,123*	$\frac{11{,}123}{800} \approx 13.904$	*13*	*14*
Subtract	*879*	$\frac{879}{800} \approx 1.099$	*1*	*1*
Multiply	*3,518*	$\frac{3{,}518}{800} \approx 4.398$	*4*	*4*
Divide	*1,563*	$\frac{1{,}563}{800} \approx 1.954$	*1*	*2*
Root	*2,917*	$\frac{2{,}917}{800} \approx 3.646$	*3*	*4*
		TOTAL	*22*	*25*

4. *Using the Jefferson Plan with a modified standard divisor of 740, complete the chart to find the standard quota and number of representatives for each state.*

State	***Population***	***Quotient***	***Standard Quota***	***Number of Reps***
Add	*11,123*	$\frac{11{,}123}{740} \approx 15.031$	*15*	*15*
Subtract	*879*	$\frac{879}{740} \approx 1.188$	*1*	*1*
Multiply	*3,518*	$\frac{3{,}518}{740} \approx 4.754$	*4*	*4*
Divide	*1,563*	$\frac{1{,}563}{740} \approx 2.112$	*2*	*2*
Root	*2,917*	$\frac{2{,}917}{740} \approx 3.942$	*3*	*3*
		TOTAL	*25*	*25*

5. *Compare the two plans. Is either of the two plans unfair? If so, what criteria/rule does it violate?*

The standard quota of Add is 13. However, the Jefferson plan assigns 15 representatives, two more than the standard quota. Therefore, the Jefferson method violates the quota rule.

6. *If a new representative were to be added (under the Jefferson method), which state would receive it using the Huntington-Hill Principle? (Hint: Calculate the Huntington-Hill number for each state.)*

Add	*Subtract*	*Multiply*	*Divide*	*Root*
$\frac{(11{,}123)^2}{15(15+1)} \approx 515{,}505$	$\frac{(879)^2}{1(1+1)} \approx 386{,}321$	$\frac{(3{,}518)^2}{4(4+1)} \approx 618{,}816$	$\frac{(1{,}563)^2}{2(2+1)} \approx 407{,}162$	$\frac{(2{,}917)^2}{3(3+1)} \approx 709{,}074$

Since Root has the greatest Huntington-Hill number, the new representative should be added to Root.

Part D: Voting Example (KEY)

Students were asked to vote on the band that would play at the annual Metric Day Dance: Binary Code (B), Wiser than Euclid (W), Get Real (G), Sum of Excellence (S), or Radical Roots (R). They used a preference ballot and came up with the following preference schedule.

Number of Voters	*18*	*12*	*10*	*9*	*4*	*2*
1st choice	*B*	*W*	*G*	*S*	*R*	*R*
2nd choice	*S*	*R*	*W*	*G*	*W*	*G*
3rd choice	*R*	*S*	*R*	*R*	*S*	*S*
4th choice	*G*	*G*	*S*	*W*	*G*	*W*
5th choice	*W*	*B*	*B*	*B*	*B*	*B*

1. *How many people voted?*

   55

2. *How many votes would be needed for a majority?*

   28

3. *Using plurality, who is the winner?*

   Binary Code, because it had 18 first choice votes.

4. *Using a Borda count, who is the winner?*

B = 5(18) + 4(0) + 3(0) + 2(0) + 1(37) = 127
W = 5(12) + 4(14) + 3(0) + 2(11) + 1(18) = 156
G = 5(10) + 4(11) + 3(0) + 2(34) + 1(0) = 162
S = 5(9) + 4(18) + 3(18) + 2(10) + 1(0) = 191
R = 5(6) + 4(12) + 3(37) + 2(0) + 1(0) = 189
So, the winner is Sum of Excellence, because they had the highest total score.

5. *Using plurality-with-elimination, who is the winner?*

Round 1: Radical Roots is eliminated because they have only 6 1st place votes.
Round 2: Sum of Excellence is eliminated because they have only 9 1st place votes.
Round 3: Wiser Than Euclid is eliminated because they have only 10 1st place votes.
Round 4: Binary Code is eliminated because they have only 18 1st place votes.
Therefore, Get Real is the winner!

6. Using the method of pair-wise comparisons, who is the winner?

Versus	***B***	***W***	***G***	***S***	***R***
B		*W*	*G*	*S*	*R*
W			*G*	*S*	*R*
G				*S*	*R*
S					*R*
R					

Since Radical Roots had the most head-to-head wins (with 4), it is the winner!

7. Do any of these methods violate the majority criterion?

Since no one received a majority of the votes, none of these methods (for this particular vote) violate the majority criterion.

8. Do any of these methods violate the condorcet criterion?

The Pair-Wise comparison test violates the condorcet criterion because Radical Roots wins in every head-to-head comparison, but it is apparent that Radical Roots does not win every election. Thus, all the methods except the Pair-Wise Comparison method violate the condorcet criterion.

Section IV: Elementary Supplement

This section has a variety of activities for elementary students that are related to voting and elections.

Weekly Activities

These weekly activities were written by Joseph A Porzio and Regina M. Mistretta. They were designed to appeal directly to students. Students may work on the activities individually or in small groups. They are meant to be open-ended questions without specific solutions, allowing students to look at themselves as the mathematical authority and developing the confidence to validate their own work.

Week One	K – 2	***Stacking and Graphing*** The teacher will supply each student with a net of a cube. Before the students cut out the net to construct their cube, they should draw or write their favorite color, pet, sport, fruit, season, and number, using one square for each choice. Then cut out, fold, and tape the net to make a cube. As a class, stack your cubes to make a bar graphs showing how each student "voted." For example, for the pet graph, all the students who chose dogs would stack their cubes next to those of the students who chose cats, and so on. What can you find out from your graphs?
	3 – 4	***The Language of Chance*** Place four yellow cubes and one blue cube in a paper bag. Is it possible to reach into the bag and pick a blue cube on the first try? How likely is it that you will pick a blue cube? Ask questions using language of chance, including such terms as *possible*, *impossible*, *likely*, *more likely*, and *less likely*. How can the language of chance be applied to the elections that are going on this month?
	5 – 6	***Studying the Polls*** Have each member of your class vote for one of the presidential candidates. On the ballot, each student should indicate whether he or she is male or female. Tally the results. Determine what percent of males voted for each candidate and what percent of females voted for each candidate. Create a circle graph for male votes and one for female votes. What conclusions can you draw from looking at your circle graphs?

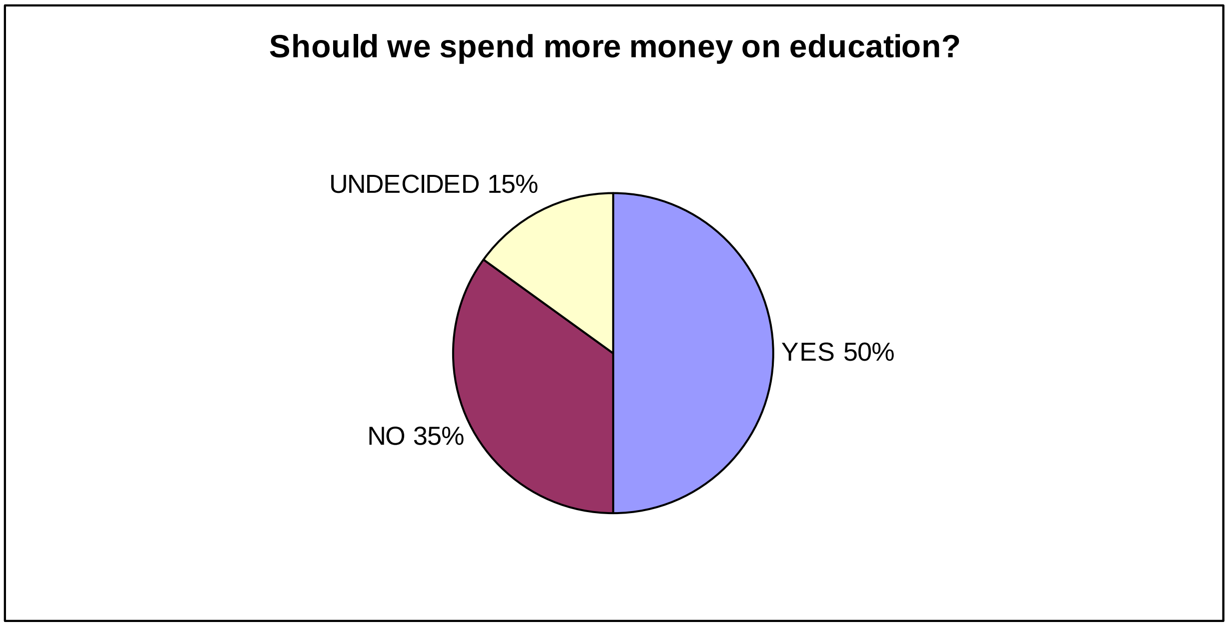
Week Two	K – 2	***Class Survey*** How do your classmates get to school? Put large loops of rope on the floor, and label the loops *by bus*, *by car*, or *bicycle* or *on foot*. Stand in the loops that describe how you get to school. Students who come to school in different ways on different days should stand in an intersection of the loops. Those who come to school in a different way should stand outside the loops. What does the survey tell you?
	3 – 4	***What shall we wear for the game?*** Each member of the class volleyball team has three uniform tops of different colors and three pairs of uniform pants of different colors. Draw pictures of each of the pieces of clothing, and then use your pictures to help you find the number of different uniforms that the team can wear for their next game. How can our presidential candidates use this method to determine how to campaign in three different states on three different days?
	5 – 6	***Electoral Votes*** The president of the United States is elected by the electoral college, which consists of 538 electors, not by popular vote. The candidate who wins the popular vote in a state usually wins the electoral votes from that state, but the electors actually determine who will be president. Visit the web site www.nara.gov/fedreg/elctcoll/votebyst.html to find out the number of electoral votes allotted to each state. What is the fewest number of states that a candidate would need to win to be elected president? List the states that the candidate must win in this type of victory.

Week Three	K – 2	***What is your prediction?*** Put three different-colored cubes or chips in a bag. Predict what results you will see if you pick a cube from the bag, record its color, and replace it. Perform this experiment thirty times, keeping track of your results. Tally the data on a chart. How did your results compare with your prediction?
	3 – 4	***Surveys and Attitudes*** Make a survey having at least three questions and at least four choices for each question. For example, "Which do you prefer: hamburgers, hot dogs, pizza, or spaghetti?" Survey your classmates, and tally, organize, display, and graph the collected data. Predict how students in other classes might respond to the same issues. Repeat your survey with another class. Compare and interpret the results.
	5 – 6	***Census 2010*** The United States Census Bureau collects data about everyone living in the United States. Participating in the census is a civic duty and is one way for people to get involved in the democratic process. Visit the Census Bureau at factfinder.census.gov to research the population of each of our fifty states. Determine which state is the most densely populated. What other issues in society are affected by the census that is conducted every ten years?
Week Four	K – 2	***November Elections*** Choose an issue that the class can vote on, such as whether recess should be in the morning or in the afternoon. Display the vote results on a chart. Graph the votes by coloring squares on a bar graph. Make sure that you give your graph a title and labels. How does the graph display the results of the vote?
	3 – 4	***Girls versus Boys*** Have each member of your class vote for one of the presidential candidates. On the ballot, each student should indicate whether he or she is male or female. Create a double-bar graph that shows the results. For example, how many girls and how many boys voted for candidate 1? What conclusions can you draw from looking at your double-bar graph? Discuss with your classmates possible reasons for your conclusions.
	5 – 6	***Rank the Issues*** During a campaign, many issues, such as crime, job security, budget, health care, education, and taxes are debated. Have your classmates rank these issues in order of importance to them. Let 1 stand for the most important issue and 5, the least. Tally your results, and decide on some method to display the results. Compare your display with those of your classmates. Discuss the benefits of each method of displaying results.

POLLS #1 **Name** ___________________________

A poll is a survey. It is a way to get information about how people feel about the issues. The people who take polls are called pollsters. They ask questions to get opinions. These opinions can suggest how people will vote in an election.

Suppose 20 people have been asked the question, “Should we spend more money on education?” Here are the results. Answer the following questions.



1. What percent of those polled thought we SHOULD spend more money on education?

2. What percent thought we SHOULD NOT?

3. What percent were undecided?

4. How could this poll effect what the candidates say about how we should spend money on education?

POLLS #2 **Name** ___________________________

A poll is a survey. It is a way to get information about how people feel (opinions) about the issues. Polls are an important source of information to a campaign. Polls are often taken by political parties, newspapers, TV networks, interest groups and independent pollsters, such as Harris. These opinions can suggest how people will vote in an election.

Find the results of a poll in the newspaper. Clip it out and answer as many of the following questions as you can from the information provided in the article.

1. Who sponsored the poll?

2. What questions were asked? Poll results can be controlled to some degree if the questions were slanted.

3. Who was interviewed? Were the people picked at random, or were they chosen to be a fair representation of the population?

4. How many people were interviewed? No matter how well a poll is done, there is always a margin of error. The fewer the number of people interviewed, the larger the margin of error.

5. How many people were undecided? When people do decide, the results could be very different.

6. When was the poll taken? Polls often show how many people feel on a certain day. They may, and often do, change their minds.

7. What impact do you think the results of the poll will have on the election?

On Your Own

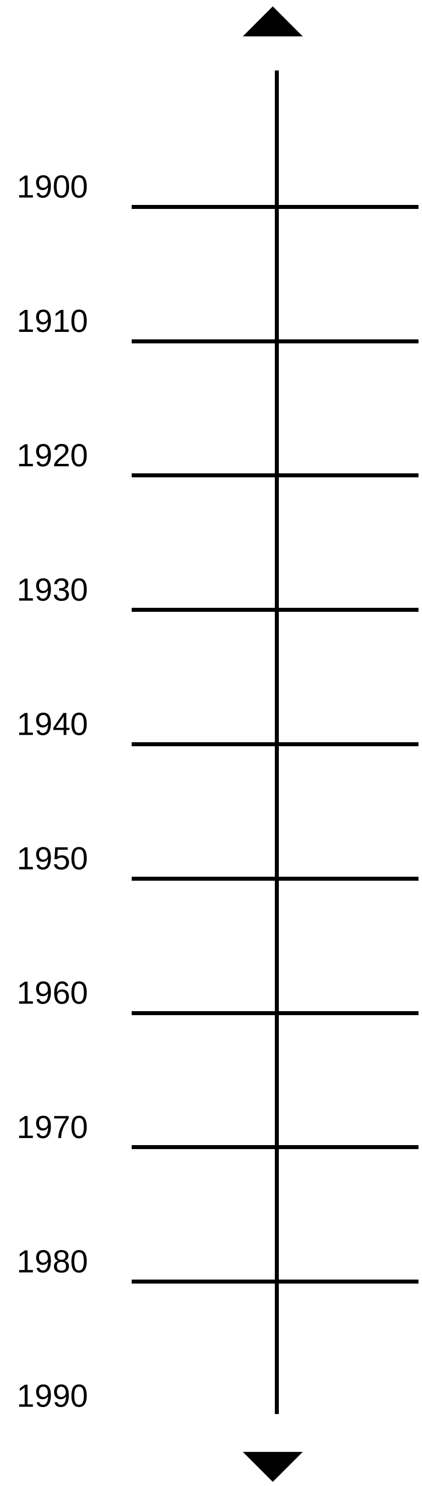
Identify an issue that is important to your class. Take a poll of seven members of the class. Based on the results, project how the whole class will vote.

Have your class vote on the issue. Compare the class vote with the poll results. Do you think the poll had an effect on the outcome of the vote? Why?

Presidential Lineup **Name** ____________________________

Here are the names of 10 men who were elected President of the United States since 1900. The year that each one took office is also shown. Find the approximate location of each year on the time line, and write the letter for each President in the correct place.

A.	Calvin Coolidge	1923
B.	John F. Kennedy	1961
C.	Woodrow Wilson	1913
D.	Franklin Roosevelt	1933
E.	George H. W. Bush	1989
F.	Theodore Roosevelt	1901
G.	Jimmy Carter	1977
H.	Harry Truman	1945
I.	Ronald Reagan	1981
J.	Dwight Eisenhower	1953



Campaign Collection Name ___________________

After a person chooses to run in an election, the campaign starts. Everything the candidate does to get votes is part of that candidate's campaign. Some examples of parts of a campaign: speeches, bumper stickers, ads on radios, ads on television, ads in newspapers, ads in magazines, a catchy slogan or saying, gifts to give away with the candidates name on them like buttons or posters, and lots of other things.

Sometimes during the campaign the candidate will say good things about himself or herself. But at other times, a candidate will spend the time and money to say bad things about the other person. This is often called 'negative campaigning.' Imagine that you were running in this election. Do you think you'd get more votes by telling people what's great about yourself or by telling them bad stuff about the other person?

Go on a Campaign Scavenger Hunt

Here are the parts of the campaign that you might hear or see between now and Election Day. Keep your score as you find each item and, on Election Day, find out which person in your class scored the most points. Under each item, write a description of what you saw or heard and where it was. If you can bring the item into school to share, you'll get five extra points for each one you bring.

A bumper sticker	20 points	
A street sign	20 points	
A campaign slogan	20 points	
The name of a political party	20 points	
A flyer that was left on a car, at your house, or anywhere else	50 points	
A giveaway (such as a pen or a button) with the candidate's name on it	50 points	
A radio/television commercial (you must tell what it said to earn the points)	50 points	
A newspaper/magazine ad	20 points	
A new thing you learned about the candidate	20 points	
Total Points I Earned		

Election Math Name ______________________

These problems are about elections. Read each of them carefully. Look for keywords to help you know how to work the problem. Show all your work!

1. Billy and Parker were running for President of the Student Council. Billy got 183 votes and Parker got 233 votes. How many people voted all together in this election?

2. Jennifer, Melody, and Sandy ran for President of the Computer Club. Jennifer received 32 votes, Sandy received 67 votes, and Melody received 71 votes. How many people voted all together in this election?

3. The Presidential election is held every four years. The election this year will be held in November 2008. In what year was the last election held?

4. Mrs. Glynn and Mr. Taylor were running for a position on the school board. Mrs. Glynn received 864 votes and Mr. Taylor received 575 votes. How many more votes to Mrs. Glynn receive than Mr. Taylor?

5. George and Eddie were running for mayor of Golftown, USA. George received 801 votes and Eddie received 729 votes. How many more votes did George receive than Eddie?

Compare the Ballots **Name** ____________________

Have students vote on Ballot One. Collect and tally those ballots. Then have students vote on Ballot Two. Collect and tally those ballots. Compare the results of the two different ballots. How did the wording affect the outcome?

☑ BALLOT ONE

On the following issues, vote YES or NO.

YES	NO	
☐	☐	1. Recess
☐	☐	2. Harry Potter
☐	☐	3. School lunch
☐	☐	4. Summer vacation

☑ BALLOT TWO

On the following issues, vote YES or NO.

YES	NO	
☐	☐	1. Recess will now be 30 minutes longer, and everyone will stay inside.
☐	☐	2. Only grown-ups are allowed to read Harry Potter books.
☐	☐	3. From now on, we will have macaroni and cheese every day for school lunch.
☐	☐	4. Instead of having a long summer vacation, we should have a short winter vacation and a short summer vacation.

Section V: High School Supplement

Will the Best Candidate Win?

This grades 9-12 lesson plan is based upon an article from the January 2000 edition of Mathematics Teacher Journal. The following activities allow students to explore alternative voting methods. Students discover what advantages and disadvantages each method offers and also see that each fails, in some way, to satisfy some desirable properties.

Learning Objectives

Students will:

- see connections between mathematics and other disciplines such as government, history, ethics, and sports
- develop skills in mathematical reasoning and apply those skills to everyday situations
- learn about various voting methods, ways in which these methods can be manipulated to achieve certain outcomes, and the impossibility of fair elections when more than two alternatives are available

Materials

- Student Activity Sheets (See end of lesson plan for these sheets!)

Instructional Plan

Students think that they are familiar with the concept of voting. After all, they have heard about governmental elections, Academy Award voting, and the ranking of the football teams. They have probably participated in club and school elections. Yet if you ask students about voting methods, most can describe only one voting method, namely, plurality. Not only have they never questioned its fairness, but considering new methods is new to them.

"The plurality method can produce a winner who is liked least."

The following activities allow students to explore alternative voting methods. They discover what advantages and disadvantages each method offers and also see that each fails, in some way, to satisfy some desirable properties. They are particularly surprised to discover that the plurality method can produce a winner who is liked least by a majority of the voters.

In addition, students look at how elections can be manipulated. One extension involves discussing Arrow's impossibility theorem, which states that it is impossible for a voting system to satisfy all the desirable features. Although these activities, including the statement and proof of Arrow's theorem, require only basic arithmetic, they allow students to engage in high-level mathematical thinking.

Note: *This activity lends itself easily to interdisciplinary instruction. Current events on the national, local, or school level can be incorporated into the project. If students are involved in making such group decisions as choosing a class gift or service project, selecting a time to hold an event, or arranging for refreshments, relevant activities can be substituted for the activities on the worksheets.*

Directions

To introduce the topic and to familiarize students with the table format, I recommend the following large group activity. Have the students suggest activities or destinations for a hypothetical class trip, and write the first three suggestions on the chalkboard. Ask each student to list her or his first, second, and third choices on a piece of paper, permitting no tied rankings.

Suppose that your students suggested archery (A), biking (B), and canoeing (C). You would then create a table, similar to **table 1**, where the columns represent all the possible preference lists.

Table 1						
Example Preference Table						
First Choice	A	A	B	B	C	C
Second Choice	B	C	A	C	A	B
Third Choice	C	B	C	A	B	A

At the top of each column, write the number of students who ranked the options in the order given. Ask students which alternative wins and how they determined the winner. If students generate only one method of tallying the winner, ask them to think of a different way.

Use the activity sheets below to help students explore different voting methods. Each activity sheet is designed for groups of three to four students.

Student Activity Sheets

Sheet 1A & 1B: The Plurality Method and Other Voting Systems

Sheet 2: Strategic Voting

Sheet 3: Tournament Digraphs and Condorcet Winners

Using Sheet 1:

The Plurality Method and Other Voting Systems
Part I (Questions 1-5)

Plurality voting is the method most familiar to students. In this method, each member is given one vote and the option that receives the most votes wins. Variations of the plurality method are used in choosing state representatives and senators, ratifying proposals, and selecting Academy Award winners. When a candidate receives more than 50 percent of the vote, including situations in which only two candidates are being considered, then plurality does produce a preferred candidate. However, in many situations, plurality may not produce a clear preference.

Since students are comfortable with the plurality method, they can usually complete this activity sheet in small groups. They are often surprised to find that skiing comes in both first and last and often mention their surprise in their responses to **question 3.** A reasonable answer for **question 4** would be the speed and ease of the plurality method. If students get stuck on **question 5**, ask them to think how winners are determined in sports tournaments or when many alternatives are available.

Note: A nice follow-up to this activity sheet is a discussion about variations of the plurality method including runoff elections, the electoral college, and two-thirds majority. Another variation of plurality, the Hare system of voting, involves a series of elections. At each stage the option, or options, with the least number of votes are eliminated from future ballots. The voters who originally voted for the eliminated option vote for the remaining option that they have ranked highest. I find that this point is worth emphasizing because students often tend to overlook these voters.

Part II (Questions 6-10)

The second half of this activity sheet introduces students to three voting methods: the **Hare system**, **Borda count**, and **sequential pair-wise voting**. Although the sheet gives instructions for each voting method, some student groups may need help following the directions.

Tip: With the Borda count, I show students how they can put a 0 next to third place, a 1 next to second place, and a 2 next to first place on the chart as an aid to totaling the points. A quick way for students to verify that their totals are reasonable is to determine the total number of votes, that is, three points per voter times forty voters, and verify that this result matches their total number of points.

Comparing the technique of sequential pair-wise voting with a single elimination tournament, with byes, may help students understand the method, but care should be taken to distinguish between the two. In sequential pair-wise voting, the pairings are sequential and no simultaneous pairings take place. Sketching the corresponding "tournament bracket" diagrams as a demonstration may help clarify the distinction.

Note: Ties do not occur in the problems given; however, the instructor should know how they are handled in the multi-step voting systems. With the Hare system, whenever two or more options share the least number of votes, they are eliminated at the same time. If all remaining candidates have the same number of votes, none are eliminated; they are all considered tied for the win. Whenever two candidates tie during a head-to-head contest in sequential pair-wise voting, neither is eliminated; they both continue and compete in a three-way contest with the next candidate.

Have students discuss the advantages and disadvantages of each of these voting methods, first in their small groups and then with the entire class.

Answers

Sheet 1

The Plurality Method and Other Voting Systems

6. In the first round of voting, skiing gets seventeen votes, rafting gets eleven, and caving gets twelve. Thus, rafting is eliminated. In the follow-up election, skiing gains one of rafting's votes, for a total of eighteen, whereas caving gets its original twelve votes plus ten votes from rafting, for a total of twenty two. Skiing is now eliminated, leaving caving as the winner.

7. Caving gets 2(12) + 17, or 41, points. Rafting gets 2(11) + 18, or 40, points. Skiing gets 2(17) + 5, or 39, points. Caving wins using the Borda count.

8. In the first vote, skiing gets eighteen votes to caving's twenty-two. Thus, skiing is eliminated and caving meets rafting in a head-to-head contest. This time, caving gets nineteen votes, whereas rafting gets twenty-one. Rafting wins.

9. Answers may vary; however, many students will assert that rafting gets an unfair advantage in problem 3.

10. A hint may be needed here. One possible answer is to eliminate skiing, the activity that the greatest number of voters ranked last, and then to hold runoff election between the remaining options.

Using Sheet 2:

Strategic Voting

Here students investigate how a voter or block of voters can influence the results of an election by submitting a ballot that does not represent their true preferences. Although the terminology is avoided on these sheets, each of the problems on sheet 2 demonstrates that a property known as **independence of irrelevant alternatives** does not hold for these voting methods. In other words, a losing candidate can win the election without any voters having moved the new winner ahead of the original winner in their preference lists. They may have moved other, irrelevant, candidates above or below one of these two. As an example, consider the following chart of preference lists:

	Number		
Ranking	2	3	4
First Choice	A	B	C
Second Choice	B	A	A
Third Choice	C	C	B

Candidate C wins using the plurality method. However, if the two people represented by the first column switched the positions of A and B in their preference list, B would win by plurality. Note that the order of B and C was not reversed. By moving B ahead of an irrelevant alternative, A, in two preference lists, B was able to win.

Question 1 on sheet 2 is adapted from *Introductory Graph Theory* (Chartrand 1985, 168).

Sheet 2

Strategic Voting

1. An editor who was voting according to his or her true preferences would probably rank his or her school first and Big City High second, or vice versa. In this problem though, students should discover that another strategy benefits the editor's school. By ranking his or her school's team first and not including Big City High among the top ten, the editor's school gets 9(9) + 10, or 91, points compared with Big City's 9(10) + 0, or 90, points. Give some credit to groups who create a tie, but point out to them that they need not include Big City High in the top ten.

2.a. If the plurality method is used, A wins, with 48 percent of the vote compared with B's 28 percent and C's 24 percent.

2.b.. If the voters in this group ranked B ahead of C, then B wins with 52% of the vote instead of A.

2.c. If the Hare system of voting is used, then C is eliminated first. In the next round, B wins with 52 percent of the vote. The last 10 percent of the voters would be most disappointed with this result. If they submitted a ballot with the ranking C, A, B, then B would be eliminated in the first round and A wins over C with 66% of the vote.

Using Sheet 3:

Tournament Digraphs and Condorcet Winners

At first, one would expect that if a candidate, called a Condorcet winner, could beat each of the other candidates in head-to-head contests, that candidate should win the election in which all candidates compete. Students are surprised to discover that this so-called Condorcet-winner criterion does not hold for the plurality method, Borda count, and Hare system. Students should be asked to explain why it does hold for sequential pair-wise voting.

In sheet 3, tournament digraphs are used to help students visualize the results of pair-wise voting. In **question 1**, most students expect candidate B to win in every method. They discover in **problem 2,** which has as its solution the digraph in 1, that although B may turn out to be the winner under each of these voting methods, they can be positive that B wins only in sequential pair-wise voting.

Problem 3 is tangential to the main topic. You may wish to omit it or use it as a take-home bonus question.

With n candidates,

$$\frac{n(n-1)}{2}$$

arrows are involved. Some students will write this expression in the form $1+2+3+...+(n-1)$. One way of deriving the first formula is by noting that n points exist and that each point has an arrow to or from each of the other $n - 1$ points. Since each arrow touches two points, the number of arrows is

$$\frac{n(n-1)}{2}$$

If students used patterns to discover the formula $1+2+3+...+(n-1)$, you can show them that

$$1 + 2 + 3 + ... + (n-1)$$

$$= \frac{(1 + 2 + 3 + ... + (n-1))}{2} + \frac{((n-1) + ... + 3 + 2 + 1)}{2}$$

$$= \frac{1}{2} + ((1 + (n - 1)) + (2 + (n - 2)) + (3 + (n - 3)) + ... + ((n-1) + 1))$$

$$= \frac{1}{2}(n - 1)\, n$$

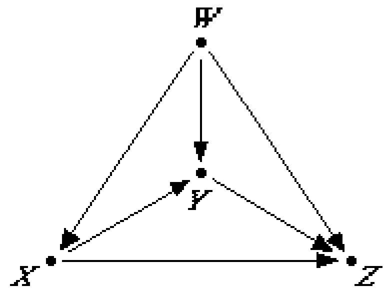
The Condorcet Winner Problem

Sheet 3 also introduces the term Condorcet winner. Problem 6 challenges students to be more creative and develop their own examples of tables of preference lists that show that the Hare system does not satisfy the Condorcet winner criterion. Make sure that students understand that their tables of preference lists for this problem must produce a Condorcet winner. Suggest that students think about how a Condorcet winner might lose an election under the Hare system of voting. Note that the Condorcet winner must be eliminated at an early stage. It cannot have a lot of first place votes. Students can later experiment with different tables of preference lists to create this situation.

Sheet 3

Tournament Digraphs and Condorcet Winners

4. The tournament digraph follows:



The exact arrangement of the vertices W, X, Y, and Z is not important. To check students' graphs, verify that exactly one arrow appears between every pair of vertices, three arrows leave W, three arrows point toward Z, and an arrow goes from X to Y.

5. By using the Borda count, W gets 3(3) + 2(2) + 2 = 15 points, X gets 3(4) + 2(3) = 18 points, Y gets 2(2) + 7 = 11 points, and Z gets 3(2) + 2(2) = 10 points, so X wins. Have students verify that they have calculated the right number of total points for each option.

6. One example is given by the following table:

2	3	3
W	X	Y
X	W	W
Y	Y	X

Here, W is a Condorcet winner that gets eliminated in the first round of Hare voting.

7. Since sequential pairwise voting involves only head-to-head contests, a Condorcet winner will win every contest it is in and hence wins the election.

Follow-Up Activities

1. Discussing Arrow's Impossibility Theorem

Now that students have discovered that each of these voting methods may not produce the expected result, a new question arises. Does a voting system exist that satisfies all desirable criteria?

Kenneth Arrow proved that not only do these four voting methods not satisfy both the Condorcet-winner criterion (CWC) and independence of irrelevant alternatives (IIA) but that it is impossible to create any voting system that does. To be more precise, any voting system that always produces at least one winner cannot satisfy both CWC and IIA.

The proof of this theorem, which can be found in Brams et al. (1996, 426-30), requires an understanding of the topics on these worksheets along with high-level mathematical reasoning. Students have difficulty understanding that if a criterion does not hold for any one table of preference lists under a particular voting method, then the voting method fails to satisfy the criterion. Other preference lists may exist for which there appears to appear no conflict.

2. Presidential Primaries

The upcoming presidential primaries offer another opportunity for extending this activity and linking it with statistics and social studies. Have students survey a sample of adults, asking them to rank the leading Republican--or Democratic, if more than three candidates run--candidates in order of preference. Ask students to create a table of preference lists illustrating their data, using each voting system studied. Students can report their sampling methods, calculations, findings, in an essay or news story.

NCTM Standards and Expectations

Number & Operations 9-12

1. Compare and contrast the properties of numbers and number systems, including the rational and real numbers, and understand complex numbers as solutions to quadratic equations that do not have real solutions.
2. Use number-theory arguments to justify relationships involving whole numbers.
3. Develop a deeper understanding of very large and very small numbers and of various representations of them.

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- Chartrand, Gary. *Introductory Graph Theory*, New York: Dover publications, 1985.

Sources for Materials and Examples:

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Is Democracy Fair? The Mathematics of Voting and Apportionment by Leslie Johnson Nielsen and Michael de Villiers, Key Curriculum Press, ISBN: 1-55953-277-7

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